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Mean Convergence of Lagrange Interpolation, II*

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The purpose of the paper is to investigate weighted L^p convergence of Lagrange interpolation taken at the zeros of Hermite polynomials. It is shown that if a continuous function satisfies some growth conditions, then the corresponding Lagrange interpolation process converges in every L^p $(1 < p < \infty)$ provided that the weight function is chosen in a suitable way.

Let $\{h_n\}_{n=0}^{\infty}$ denote the system of orthonormalized Hermite polynomials, and let $x_{1n} > x_{2n} > \cdots > x_{nn}$ be the zeros of h_n. Then for a given function f the Lagrange interpolation polynomial $L_n(f)$ corresponding to h_n is defined to be the unique algebraic polynomial of degree at most $n-1$ which satisfies

$$L_n(f, x_{kn}) = f(x_{kn}) \qquad (k = 1, 2, \ldots, n).$$

It is well known that $L_n(f)$ can be written in the form

$$L_n(f, x) = \sum_{k=1}^{n} f(x_{kn})\, l_{kn}(x),$$

where the fundamental polynomials l_{kn} are defined by

$$l_{kn}(x) = \frac{h_n(x)}{h_n'(x_{kn})(x - x_{kn})} = \sqrt{\frac{n}{2}}\,\lambda_{kn} h_{n-1}(x_{kn}) \frac{h_n(x)}{(x - x_{kn})}$$

[8, p. 48]. Here λ_{kn} $(k = 1, 2, \ldots, n)$ denote the Christoffel numbers of the corresponding Gauss–Jacobi quadrature formula.

The purpose of this paper is to investigate weighted L^p convergence properties of $L_n(f)$. For the sake of brevity we do not intend to discuss the history of this problem. We refer the interested reader to [1, 4, 5]. Our main result is the following:

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THEOREM 1. *Let f be a continuous function defined on the real line. Assume that f satisfies*

$$\lim_{x\to\infty} f(x)(1+|x|)\,e^{-x^2/2}=0. \tag{1}$$

Then

$$\lim \int_{-\infty}^{\infty} [|f(x)-L_n(f,x)|\,e^{-x^2/2}|^p\,dx=0$$

holds for every $p>1$.

In order to justify the choice of the weight function in Theorem 1, we will also prove

THEOREM 2. *Let $w(\geqslant 0)\in L^1(\mathbb{R})$ and $0<p<\infty$ be given. Suppose that for every continuous function f vanishing outside a finite interval*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x)-L_n(f,x)|^p\,w(x)\,dx=0. \tag{2}$$

Then

$$\int_{-\infty}^{\infty} \left[\frac{e^{x^2/2}}{1+|x|}\right]^p w(x)\,dx<\infty.$$

The proofs of these two theorems require several auxiliary results. First we will prove these results and then we will be able to verify our main theorems. In the following, every positive constant independent of the variables in consideration will be denoted by "const."

LEMMA 3. *There exists a number $c>0$ such that for every polynomial R_n of degree n*

$$\int_{-\infty}^{\infty} |R_n(x)|\,e^{-x^2}\,dx \leqslant \text{const} \int_{-c\sqrt{n}}^{c\sqrt{n}} |R_n(x)|\,e^{-x^2}\,dx.$$

This lemma was proved by Freud in [3].

LEMMA 4. *The inequality*

$$\sum_{k=1}^{n} \lambda_{kn} e^{x_{kn}^2} \leqslant e\sqrt{\pi(2n+1)}$$

holds for every $n=1,2,\ldots$.

Proof. Let $0 < a < 1$. Then the function e^{ax^2} is such that all its derivatives of even order are non-negative. Thus by Markoff's theorem [8, p. 378]

$$\sum_{k=1}^{n} \lambda_{kn} e^{ax_{kn}^2} \leqslant \int_{-\infty}^{\infty} e^{(a-1)x^2}\, dx = \sqrt{\frac{\pi}{1-a}}.$$

Setting $a = 1 - 1/(2n+1)$ we obtain

$$\sum_{k=1}^{n} \lambda_{kn} e^{x_{kn}^2} \leqslant \sqrt{\pi(2n+1)} \max_{1\leqslant k\leqslant n} e^{x_{kn}^2/(2n+1)}.$$

Now the lemma follows from the inequality $x_{kn}^2 \leqslant 2n+1$ $(k = 1, 2,..., n)$ [8, p. 129].

LEMMA 5. *Let $\sigma \in (0, \sqrt{2})$ be given. Then for every polynomial P_m of degree at most m the inequality*

$$\sum_{|x_{kn}|\leqslant\sigma\sqrt{n}} \lambda_{kn} |P_m(x_{kn})| \frac{e^{x_{kn}^2/2}}{1+|x_{kn}|} \leqslant \text{const} \left(1 + \sqrt{\frac{m}{n}}\right) \int_{-\infty}^{\infty} |P_m(x)| \frac{1}{1+|x|} e^{-x^2/2}\, dx. \tag{3}$$

Proof. First we will show that

$$\sum_{|x_{kn}|\leqslant\sigma\sqrt{n}} \lambda_{kn} |P_m(x_{kn})|\, e^{x_{kn}^2/2} \leqslant \text{const} \left(1 + \sqrt{\frac{m}{n}}\right) \int_{-\infty}^{\infty} |P_m(x)|\, e^{-x^2/2}\, dx. \tag{4}$$

In order to prove (4) let us note that

$$|P_m(x_{kn})| \leqslant \min_{x_{k+1,n}\leqslant t\leqslant x_{k-1,n}} |P_m(t)| + \int_{x_{k+1,n}}^{x_{k-1,n}} |P'_m(x)|\, dx.$$

Thus, by the Markoff–Stieltjes inequality

$$\lambda_{kn} \leqslant \int_{x_{k+1,n}}^{x_{k-1,n}} e^{-t^2}\, dt \tag{5}$$

[8, p. 50], we have

$$\begin{aligned}
&\sum_{|x_{kn}|\leqslant\sigma\sqrt{n}} \lambda_{kn} |P_m(x_{kn})|\, e^{x_{kn}^2/2} \\
&\quad\leqslant \sum_{|x_{kn}|\leqslant\sigma\sqrt{n}} e^{x_{kn}^2/2} \int_{x_{k+1,n}}^{x_{k-1,n}} |P_m(t)|\, e^{-t^2}\, dt \\
&\qquad + \sum_{|x_{kn}|\leqslant\sigma\sqrt{n}} \lambda_{kn}\, e^{x_{kn}^2/2} \int_{x_{k+1,n}}^{x_{k-1,n}} |P'_m(x)|\, dx.
\end{aligned}$$

Inequalities (5) and

$$x_{k-1,n} - x_{k+1,n} \leqslant \text{const } n^{-1/2} \qquad (|x_{kn}| \leqslant \sigma\sqrt{n})$$

[2, p. 180] imply that

$$e^{x_{kn}^2/2} \leqslant \text{const } e^{t^2/2} \qquad (x_{k+1,n} \leqslant t \leqslant x_{k-1,n})$$

and

$$\lambda_{kn} e^{x_{kn}^2/2} \leqslant \text{const } n^{-1/2} e^{-x^2/2} \qquad (x_{k+1,n} \leqslant x \leqslant x_{k-1,n})$$

for $|x_{kn}| \leqslant \sigma\sqrt{n}$. Hence

$$\sum_{|x_{kn}| \leqslant \sigma\sqrt{n}} \lambda_{kn} |P_m(x_{kn})| e^{x_{kn}^2/2}$$
$$\leqslant \text{const } \left[\int_{-\infty}^{\infty} |P_m(t)| e^{-t^2/2} dt + n^{-1/2} \int_{-\infty}^{\infty} |P_m'(x)| e^{-x^2/2} dx. \qquad (6)\right.$$

Applying G. Freund's Markoff-type inequality [3] to the second integral on the right side of (6), we immediately obtain (4). Now we will prove that inequality (4) implies (3). Note, that in (3) we can assume without loss of generality that $m \geqslant n$. Let $C > 0$ be an arbitrary but fixed number. Suppose that there exists a polynomial π_m of degree at most m such that

$$\frac{1}{1+|t|} \leqslant \text{const } |\pi_m(t)| \tag{7}$$

for $|t| \leqslant \sigma\sqrt{n}$ and

$$|\pi_m(t)| \leqslant \text{const } \frac{1}{1+|t|} \tag{8}$$

for $|t| \leqslant C\sqrt{m}$. Then by (4) and Lemma 3

$$\sum_{|x_{kn}| \leqslant \sigma\sqrt{n}} \lambda_{kn} |P_m(x_{kn})| \frac{e^{x_{kn}^2/2}}{1+|x_{kn}|}$$
$$\leqslant \text{const } \left(1 + \sqrt{\frac{m}{n}}\right) \int_{-C_1\sqrt{m}}^{C_1\sqrt{m}} |P_m(x)| \, |\pi_m(x)| \, e^{-x^2/2} dx$$

with some constant $C_1 > 0$. Now, if C is chosen so that $C > C_1$, then we can apply (8) and the lemma follows. Thus the lemma will be proved if we can construct a polynomial π_m such that inequalities (7) and (8) hold. Since $\sigma < \sqrt{2}$, $n \leqslant m$ and the function $(1+|t|)^{-1}$ is exactly of the same size as

$(1+t^2)^{-1/2}$, it will be enough to show that for every $C>2$ there exist a polynomial π_m such that

$$0 < M_1 \, |\pi_m(t)| \leqslant (1+t^2)^{-1/2} \leqslant M_2 \, |\pi_m(t)| \tag{9}$$

for $|t| \leqslant C\sqrt{m}$, where M_1 and M_2 are independent of m and t. Since the second derivative of $(1+t^2)^{-1/2}$ in absolute value is bounded by 4, we can apply Jackon's theorem [8, p. 6] to conclude that there exists a polynomial ρ_m such that

$$|(1+t^2)^{-1/2} - \rho_m(t)| \leqslant \text{const}\,\frac{1}{m}$$

for $|t| \leqslant \sqrt{m}$. Therefore, if m is big enough then

$$|(1+t^2)^{-1/2} - \rho_m(t)| \leqslant \tfrac{1}{2}(1+t^2)^{-1/2}$$

for $|t| \leqslant \sqrt{m}$, that is,

$$\tfrac{2}{3}\,|\rho_m(t)| \leqslant (1+t^2)^{-1/2} \leqslant 2\,|\rho_m(t)| \qquad (|t| \leqslant \sqrt{m}).$$

Thus, if we put $\pi_m(t) = \rho_m(t/C)$, then π_m satisfies (9) since for fixed values of C the functions $(1+t^2)^{-1/2}$ and $(1+t^2/C^2)^{-1/2}$ are of the same size.

LEMMA 6. *Let $a>0$ be fixed and let $\{f_n\}$ be a sequence of functions such that $f_n(x)=0$ for $|x| \leqslant a\sqrt{n}$ and*

$$|f_n(x)| \leqslant \frac{e^{x^2/2}}{1+|x|}$$

for $x \in \mathbb{R}$. Then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |L_n(f_n, x)\, e^{-x^2/2}|^p \, dx = 0 \tag{10}$$

for every $p>1$.

Proof. Applying Schwarz' inequality we obtain

$$\begin{aligned} L_n(f_n, x)^2 &\leqslant \sum_{|x_{kn}|>a\sqrt{n}} f_n^2(x_{kn})\,\lambda_{kn} \sum_{k=1} \frac{l_{kn}^2(x)}{\lambda_{kn}} \\ &\leqslant \frac{1}{a^2 n} \sum_{k=1}^{n} \lambda_{kn} e^{x_{kn}^2} \sum_{k=1}^{n} \frac{l_{kn}^2(x)}{\lambda_{kn}}. \end{aligned}$$

Hence by Lemma 4 and

$$\sum_{k=1}^{n} \frac{l_{kn}^2(x)}{\lambda_{kn}} \leqslant \text{const}\, \sqrt{n}\, e^{x^2} \qquad (x \in \mathbb{R})$$

[2, p. 181] the inequality

$$|L_n(f_n, x)\, e^{-x^2/2}| \leqslant \text{const} \tag{11}$$

holds independently of x and n. Now, if $p \geqslant 2$, then (11) implies

$$\int_{-\infty}^{\infty} |L_n(f_n, x)\, e^{-x^2/2}|^p\, dx \leqslant \text{const} \int_{-\infty}^{\infty} |L_n(f_n, x)\, e^{-x^2/2}|^2\, dx. \tag{12}$$

Using the Gauss–Jacobi quadrature formula we get

$$\begin{aligned}\int_{-\infty}^{\infty} |L_n(f_n, x)\, e^{-x^2/2}|^2\, dx &= \sum_{|x_{kn}| > a\sqrt{n}} f_n^2(x_{kn})\, \lambda_{kn} \\ &\leqslant \frac{1}{a^2 n} \sum_{k=1}^{n} \lambda_{kn} e^{x_{kn}^2}.\end{aligned}$$

Thus by Lemma 4

$$\int_{-\infty}^{\infty} |L_n(f_n, x)\, e^{-x^2/2}|^2\, dx \leqslant \text{const}\, n^{-1/2} \tag{13}$$

which together with (12) proves (10) for $p \geqslant 2$. Now let $1 < p < 2$. First we will show that for every fixed $c > 0$

$$\lim_{n\to\infty} \int_{-c\sqrt{n}}^{c\sqrt{n}} |L_n(f_n, x)\, e^{-x^2/2}|^p\, dx = 0. \tag{14}$$

Since $p < 2$, we have by Hölder's inequality

$$\begin{aligned}&\int_{-c\sqrt{n}}^{c\sqrt{n}} |L_n(f_n, x)\, e^{-x^2/2}|^p\, dx \\ &\qquad \leqslant \left[\int_{-c\sqrt{n}}^{c\sqrt{n}} |L_n(f_n, x)\, e^{-x^2/2}|^2\, dx\right]^{p/2} |2c\sqrt{n}|^{(2-p)/2}.\end{aligned}$$

Consequently by (13)

$$\int_{-c\sqrt{n}}^{c\sqrt{n}} |L_n(f_n, x)\, e^{-x^2/2}|^p\, dx \leqslant \text{const}\, n^{(1-p)/2},$$

which implies (14) for $1 < p < 2$. Thus (10) will be proved for $1 < p < 2$ if we can show that for every fixed $c > \sqrt{2}$

$$\lim_{n\to\infty} \int_{|x| > c\sqrt{n}} |L_n(f_n, x)\, e^{-x^2/2}|^p\, dx = 0.$$

Using

$$L_n(f_n, x) = \sqrt{\frac{n}{2}}\, h_n(x) \sum_{|x_{kn}| > a\sqrt{n}} \lambda_{kn} h_{n-1}(x_{kn}) \frac{f_n(x_{kn})}{x - x_{kn}}$$

we obtain

$$L_n(f_n, x)^2 \leqslant \text{const } n \frac{h_n^2(x)}{x^2} \sum_{|x_{kn}| > a\sqrt{n}} \lambda_{kn} f_n^2(x_{kn}) \sum_{k=1}^{n} \lambda_{kn} h_{n-1}^2(x_{kn})$$

for $|x| > c\sqrt{n}$ since $|x_{kn}| < \sqrt{2n+1}$ $(k = 1, 2,..., n)$ [8, p. 129]. Thus by Lemma 4

$$|L_n(f_n, x)\, e^{-x^2/2}|^p\, dx \leqslant \text{const } n^{p/4} \left| \frac{h_n(x)\, e^{-x^2/2}}{x} \right|^p \qquad (|x| > c\sqrt{n}).$$

Integrating this inequality and applying Hölder's inequality we get

$$\begin{aligned} &\int_{|x| > c\sqrt{n}} |L_n(f_n, x)\, e^{-x^2/2}|^p\, dx \\ &\quad \leqslant \text{const } n^{p/4} \int_{c\sqrt{n}}^{\infty} \left| \frac{h_n(x)\, e^{-x^2/2}}{x} \right|^p dx \\ &\quad \leqslant \text{const } n^{p/4} \left[\int_{-\infty} h_n^2(x)\, e^{-x^2}\, dx \right]^{p/2} \left[\int_{c\sqrt{n}}^{\infty} x^{2p/(p-2)}\, dx \right]^{(2-p)/2} \\ &\quad = \text{const } n^{(1-p)/2}, \end{aligned}$$

which proves our assertion since $1 < p < 2$.

Lemma 7. *Let $\{\varphi_n\}$ be a sequences of functions such that $\varphi_n(x) = 0$ for $|x| \geqslant \frac{1}{2}\sqrt{n}$ and*

$$|\varphi_n(x)| \leqslant \frac{e^{x^2/2}}{1 + |x|} \tag{15}$$

for $x \in \mathbb{R}$. Then

$$\lim_{n\to\infty} \int_{|x| \geqslant \sqrt{n}} |L_n(\varphi_n, x)\, e^{-x^2/2}|^p\, dx = 0$$

for every $p > 1$.

Proof. It follows from

$$L_n(\varphi_n, x) = \sqrt{\frac{n}{2}}\, h_n(x) \sum_{|x_{kn}| \leqslant (1/2)\sqrt{n}}' \lambda_{kn} h_{n-1}(x_{kn}) \frac{\varphi_n(x_{kn})}{x - x_{kn}}$$

that the inequality

$$\begin{aligned} &\int_{|x| \geqslant \sqrt{n}} |L_n(\varphi_n, x)\, e^{-x^2/2}|^p \, dx \\ &\quad \leqslant \text{const}\, n^{p/2} \left[\sum_{|x_{kn}| \leqslant (1/2)\sqrt{n}}' \lambda_{kn} |h_{n-1}(x_{kn})|\, |\varphi_n(x_{kn})| \right]^p \\ &\quad \times \int_{\sqrt{n}}^{\infty} \left| \frac{h_n(x)\, e^{-x^2/2}}{x} \right|^p dx \end{aligned} \tag{16}$$

holds. First let us examine the expression in brackets. By Lemma 5 and (15) we have

$$\begin{aligned} &\sum_{|x_{kn}| \leqslant (1/2)\sqrt{n}}' \lambda_{kn} |h_{n-1}(x_{kn})|\, |\varphi_n(x_{kn})| \\ &\quad \leqslant \sum_{|x_{kn}| \leqslant (1/2)\sqrt{n}}' \lambda_{kn} |h_{n-1}(x_{kn})| \frac{e^{x_{kn}^2/2}}{1 + |x_{kn}|} \\ &\quad \leqslant \text{const} \int_{-\infty}^{\infty} |h_{n-1}(x)| \frac{e^{-x^2/2}}{1 + |x|}\, dx. \end{aligned}$$

Applying the inequality

$$|h_{n-1}(x)|\, e^{-x^2/2} \leqslant \text{const}\, n^{-1/4} \qquad (|x| \leqslant \sqrt{n}) \tag{17}$$

[8, p. 201] we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{|h_{n-1}(x)|\, e^{-x^2/2}}{1 + |x|}\, dx &\leqslant \text{const}\, n^{-1/4} \int_{-\sqrt{n}}^{\sqrt{n}} \frac{1}{1 + |x|}\, dx \\ &\quad + \left[\int_{|x| \geqslant \sqrt{n}} h_{n-1}^2(x) e^{-x^2} dx \cdot \int_{|x| \geqslant \sqrt{n}} (1 + |x|)^2 \, dx \right]^{1/2} \\ &\leqslant \text{const}\, n^{-1/4} \log n \end{aligned}$$

so that

$$\sum_{|x_{kn}| \leqslant (1/2)\sqrt{n}}' \lambda_{kn} |h_{n-1}(x_{kn})|\, |\varphi_n(x_{kn})| \leqslant \text{const}\, n^{-1/4} \log n. \tag{18}$$

Now let us estimate the integral on the right side of (16). If $p > 2$ then

$$\int_{\sqrt{n}}^{\infty} \left| \frac{h_n(x)\, e^{-x^2/2}}{x} \right|^p dx \leqslant \text{const} \cdot n^{(1-p)/2} \tag{19}$$

since $|h_n(x)|\, e^{-x^2/2} \leqslant \text{const}$ for $x \in \mathbb{R}$ [8, p. 242]. If $1 < p < 2$ then by Hölder's inequality

$$\int_{\sqrt{n}}^{\infty} \left| \frac{h_n(x)\, e^{-x^2/2}}{x} \right|^p dx \leqslant \left[\int_{-\infty}^{\infty} h_n^2(x)\, e^{-x^2}\, dx \right]^{p/2} \left[\int_{\sqrt{n}}^{\infty} x^{2p/(p-2)}\, dx \right]^{(2-p)/2}$$
$$= \text{const}\; n^{(2-3p)/4}. \tag{20}$$

We obtain from inequalities (16), (18), (19) and (20) that

$$\int_{|x| \geqslant \sqrt{n}} |L_n(\varphi_n, x)\, e^{-x^2/2}|^p\, dx \leqslant \text{const}|\log n|^p \begin{cases} n^{(2-p)/4} & (p > 2) \\ n^{(1-p)/2} & (1 < p < 2), \end{cases}$$

which proves the lemma.

LEMMA 8. *Let $\varepsilon > 0$ be given. Let $\{\psi_n\}$ be a sequence of functions such that $\psi_n(x) = 0$ for $|x| > \sqrt{n}$ and*

$$|\psi_n(x)| \leqslant \varepsilon \frac{e^{x^2/2}}{1 + |x|} \tag{21}$$

for $x \in \mathbb{R}$. Then

$$\limsup_{n \to \infty} \int_{-\sqrt{n}}^{\sqrt{n}} |L_n(\psi_n, x)\, e^{-x^2/2}|^p\, dx \leqslant \text{const}\; \varepsilon^p$$

for every $p > 1$.

Proof. Let $S_n(g, x)$ denote the nth partial sum of the Fourier expansion of some function g in the Hermite polynomials $\{h_k\}$. Let G be defined by

$$G(x) = \frac{e^{x^2/2}}{1 + |x|}. \tag{22}$$

First we will show that

$$\left[\int_{-\sqrt{n}}^{\sqrt{n}} |L_n(\psi_n, x)\, e^{-x^2/2}|^p\, dx \right]^{1/p}$$
$$\leqslant \text{const}\; \varepsilon \cdot \sup_{\|\beta\|_\infty \leqslant 1} \left[\int_{-\sqrt{n}}^{\sqrt{n}} |S_n(\beta G, x)\, e^{-x^2/2}|^p\, dx \right]^{1/p}, \tag{23}$$

where $\|\cdot\|_\infty$ denotes the usual L^∞ norm. To prove (23) let us introduce a function g_n defined by

$$g_n(x) = \operatorname{sign}[L_n(\psi_n, x)]\, |L_n(\psi_n, x)|^{p-1}\, 1_{\sqrt{n}}(x)\, e^{(1-p/2)x^2}, \tag{24}$$

where $1_{\sqrt{n}}(x)$ is the characteristic function of the interval $[-\sqrt{n}, \sqrt{n}]$. Then

$$\int_{-\sqrt{n}}^{\sqrt{n}} |L_n(\psi_n, x)\, e^{-x^2/2}|^p\, dx = \int_{-\infty}^{\infty} L_n(\psi_n, x)\, g_n(x)\, e^{-x^2}\, dx.$$

Since $L_n(\psi_n, x)$ is a polynomial of degree less than n, we get

$$\int_{-\sqrt{n}}^{\sqrt{n}} |L_n(\psi_n, x)\, e^{-x^2/2}|^p\, dx = \int_{-\infty}^{\infty} L_n(\psi_n, x)\, S_n(g_n, x)\, e^{-x^2}\, dx.$$

Note that $L_n(\psi_n, x)\, S_n(g_n, x)$ is a polynomial of degree at most $2n-1$. Thus we can apply the Gauss–Jacobi quadrature formula to obtain

$$\int_{-\sqrt{n}}^{\sqrt{n}} |L_n(\psi_n, x)\, e^{-x^2/2}|^p\, dx = \sum_{k=1}^{n} L_n(\psi_n, x_{kn})\, S_n(g_n, x_{kn})\, \lambda_{kn}.$$

Because $L_n(\psi_n, x)$ interpolates ψ_n at x_{kn}, it follows from (21) that the inequality

$$\int_{-\sqrt{n}}^{\sqrt{n}} |L_n(\psi_n, x)\, e^{-x^2/2}|^p \leqslant \varepsilon \sum_{|x_{kn}| \leqslant \sqrt{n}} |S_n(g_n, x_{kn})|\, G(x_{kn})\, \lambda_{kn}$$

holds where G was defined in (22). Thus by Lemma 5

$$\int_{-\sqrt{n}}^{\sqrt{n}} |L_n(\psi_n, x)\, e^{-x^2/2}|p\, dx \leqslant \text{const}\, \varepsilon \int_{-\infty}^{\infty} |S_n(g_n, x)|\, G(x)\, e^{-x^2}\, dx. \tag{25}$$

Let β_n be defined by

$$\beta_n(x) = \operatorname{sign} S_n(g_n, x)$$

Then $\|\beta_n\|_\infty \leqslant 1$ and

$$\int_{-\infty}^{\infty} |S_n(g_n, x)|\, G(x)\, e^{-x^2}\, dx = \int_{-\infty}^{\infty} g_n(x)\, S_n(\beta_n G, x)\, e^{-x^2}\, dx.$$

Consequently by (24)

$$\int_{-\infty}^{\infty} |S_n(g_n, x)|\, G(x)\, e^{-x^2}\, dx$$
$$= \int_{-\sqrt{n}}^{\sqrt{n}} \operatorname{sign}[L_n(\psi_n, x)]\, L_n(\psi_n, x)\, e^{-x^2/2}|^{p-1}\, [S_n(\beta_n G, x)\, e^{-x^2/2}]\, dx.$$

Applying Hölder's inequality we obtain

$$\int_{-\infty}^{\infty} |S_n(g_n, x)|\, G(x)\, e^{-x^2}\, dx$$
$$\leqslant \left[\int_{-\sqrt{n}}^{\sqrt{n}} |L_n(\psi_n, x)\, e^{-x^2/2}|^p\, dx\right]^{(p-1)/p} \cdot \left[\int_{-\sqrt{n}}^{\sqrt{n}} |S_n(\beta_n G, x)\, e^{-x^2/2}|^p\, dx\right]^{1/p}.$$

This inequality combined with (25) yields (23). In order to complete the proof of the lemma we must prove that

$$\sup_{\|\beta\|_\infty \leqslant 1} \left[\int_{-\sqrt{n}}^{\sqrt{n}} |S_n(\beta G, x)\, e^{-x^2/2}|^p\, dx\right]^{1/p} \leqslant \text{const} \tag{26}$$

independently of n. Using the Christoffel–Darboux formula [8, p. 43] we get

$$S_n(\beta G, x)\, e^{-x^2/2} = \sqrt{\frac{n}{2}}\, h_n(x)\, e^{-x^2/2} \int_{-\infty}^{\infty} \frac{h_{n-1}(t)\, \beta(t)\, G(t)\, e^{-t^2}}{x-t}\, dt$$
$$- \sqrt{\frac{n}{2}}\, h_{n-1}(x)\, e^{-x^2/2} \int_{-\infty}^{\infty} \frac{h_n(t)\, \beta(t)\, G(t)\, e^{-t^2}}{x-t}\, dt,$$

where the integrals are defined in the sense of Cauchy. Since x in (26) varies between $-\sqrt{n}$ and $\sqrt{n}$, and the Hermite polynomials satisfy (17), inequality (26) will be proved if we show that

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{h_n(t)\, \beta(t)\, G(t)\, e^{-t^2}}{x-t}\, dt \right|^p dx \leqslant \text{const}\, n^{-p/4}\, \|\beta\|_\infty^p \tag{27}$$

for $n = 1, 2, \ldots$. Let us recall that te Hilbert transform is a bounded operator on L^p if $1 < p < \infty$ [7]. Thus (27) holds if

$$\int_{-\infty}^{\infty} |h_n(x)\, \beta(x)\, G(x)\, e^{-x^2}|^p\, dx \leqslant \text{const}\, n^{-p/4}\, \|\beta\|_\infty^p,$$

that is,

$$\int_{-\infty}^{\infty} \left| \frac{h_n(x)\, e^{-x^2/2}}{1+|x|} \right|^p dx \leqslant \text{const}\, n^{-p/4}. \tag{28}$$

By (17) the inequality

$$\int_{-\sqrt{n}}^{\sqrt{n}} \left| \frac{h_n(x)\, e^{-x^2/2}}{1+|x|} \right|^p dx \leqslant \text{const}\, n^{-p/4}$$

holds. Hence by (19) and (20) inequality (28) is satisfied.

Proof of Theorem 1. Let $\varepsilon > 0$ be fixed. They by (1) we can find a polynomial P such that

$$|f(x) - P(x)| \leqslant \varepsilon \frac{e^{x^2/2}}{1 + |x|}$$

for $x \in \mathbb{R}$. Thus, if $n \geqslant \deg(P)$, then

$$\begin{aligned}
&\left\{\int_{-\infty}^{\infty} [|f(x) - L_n(f, x)| e^{-x^2/2}]^p \, dx\right\}^{1/p} \\
&\quad \leqslant \left\{\int_{-\infty}^{\infty} [|f(x) - P(x)| e^{-x^2/2}]^p \, dx\right\}^{1/p} \\
&\quad\quad + \left\{\int_{-\infty}^{\infty} \left[|L_n(f - P, x)| e^{-x^2/2}\right]^p \, dx\right\}^{1/p} \\
&\quad \leqslant \varepsilon \left\{\int_{-\infty}^{\infty} (1 + |x|)^{-p} \, dx\right\}^{1/p} \\
&\quad\quad + \left\{\int_{-\infty}^{\infty} [|L_n(f - P, x)| e^{-x^2/2}]^p \, dx\right\}^{1/p}.
\end{aligned} \tag{29}$$

Now consider $L_n(f - P)$. Let 1_n denote the characteristic function of $[-\frac{1}{2}\sqrt{n}, \frac{1}{2}\sqrt{n}]$. Then we can decompose $f - P$ into

$$f - P = (f - P)\, 1_n + (f - P)(1 - 1_n) = u_n + v_n. \tag{30}$$

The function u_n satisfies the conditions of Lemmas 7 and 8. Hence

$$\limsup_{n \to \infty} \left\{\int_{-\infty}^{\infty} [|L_n(u_n, x)| e^{-x^2/2}]^p \, dx\right\}^{1/p} \leqslant \text{const } \varepsilon.$$

On the other hand, we can apply Lemma 6 to v_n to obtain

$$\limsup_{n \to \infty} \left\{\int_{-\infty}^{\infty} [|L_n(v_n, x)| e^{-x^2/2}]^p \, dx\right\}^{1/p} = 0.$$

Thus by (30)

$$\limsup_{n \to \infty} \left\{\int_{-\infty}^{\infty} [|L_n(f - P, x) e^{-x^2/2}]^p \, dx\right\}^{1/p} \leqslant \text{const } \varepsilon.$$

Using (29) we get

$$\limsup_{n \to \infty} \left\{\int_{-\infty}^{\infty} [|f(x) - L_n(f, x)| e^{-x^2/2}]^p \, dx\right\}^{1/p} \leqslant \text{const } \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the theorem follows.

Proof of Theorem 2. Let $C_0(-2,-1)$ denote the space of continuous functions on $\mathbb{R}$ with support in $[-2,-1]$. Then $L_n(f)$ can be considered a linear functional on $C_0(-2,-1)$ and by conditions (2) holds for each $f \in C_0(-2,-1)$. Hence by Theorem 10.19 of [6, p. 182]

$$\int_{-\infty}^{\infty} |L_n(f,x)|^p\, w(x)\, dx \leqslant \text{const} \max_{-2 \leqslant x \leqslant -1} |f(x)|^p \tag{31}$$

for $f \in C_0(-2,-1)$. Now for every $n=1,2,\ldots$ let us pick up a function $g_n \in C_0(-2,-1)$ such that

$$\max_{-2 \leqslant x \leqslant -1} |g_n(x)| = 1$$

and

$$g_n(x_{kn}) = \operatorname{sign} h_n'(x_{kn}).$$

Then

$$L_n(g_n,x) = h_n(x) \sum_{-2 \leqslant x_{kn} \leqslant -1} |h_n'(x_{kn})|^{-1} (x-x_{kn})^{-1}.$$

It follows from $h_n'(x) = \sqrt{2n}\, h_{n-1}(x)$ and $|h_{n-1}(x_{kn})| \leqslant \text{const}\, n^{-1/4}$ for $-2 \leqslant x_{kn} \leqslant -1$ [2, p. 181] that

$$|h_n'(x_{kn})|^{-1} \geqslant \text{const}\, n^{-1/4} \qquad (-2 \leqslant x_{kn} \leqslant -1).$$

Furthermore, we obtain from

$$\frac{\pi}{\sqrt{2n+1}} \leqslant x_{kn} - x_{k+1,n} \leqslant \frac{\pi}{\sqrt{n+1}} \qquad (-2 \leqslant x_{kn} \leqslant -1)$$

[2, p. 180] that the number of zeros of $h_n(x)$ in $[-2,-]$ is exactly of order $\sqrt{n}$. Hence

$$|L_n(g_n,x)| \geqslant \text{const}\, n^{1/4}\, |h_n(x)|\, (1+x)^{-1}$$

whenever $x > 0$. Thus by (31) we can conclude that the inequality

$$A = \limsup_{n\to\infty} \int_{-\infty}^{\infty} \left| \frac{n^{1/4} h_n(x)}{1+x} \right|^p w(x)\, dx < \infty \tag{32}$$

holds. Now let us fix $M > 0$. By Fejér's asymptotic formula for the Hermite polynomials [8, p. 200]

$$n^{1/4} h_n(x) = \text{const}\, e^{x^2/2} \cos\left[\sqrt{2n+1}\,x - \frac{n\pi}{2}\right] + o(1)$$

$(0 \leqslant x \leqslant M)$ inequality (32) implies

$$\limsup_{n\to\infty} \int_0^M \left[\frac{e^{x^2/2}}{1+x}\right]^p w(x) \left|\cos\left[\sqrt{2n+1}\,x - \frac{n\pi}{2}\right]\right|^p dx \leqslant \text{const}\, A.$$

Hence, if $m \geqslant p$ is an even integer then

$$\limsup_{n\to\infty} \int_0^M \left[\frac{e^{x^2/2}}{1+x}\right]^p w(x) |\cos(\sqrt{4n+1}\,x)|^m \, dx \leqslant \text{const}\, A.$$

Since $(\cos x)^m$ is a trigonometric polynomial with non zero constant term, applying the Riemann–Lebesgue lemma we obtain

$$\int_0^M \left[\frac{e^{x^2/2}}{1+x}\right]^p w(x)\, dx \leqslant \text{const}\, A.$$

Letting $M \to \infty$ the inequality

$$\int_0^\infty \left[\frac{e^{x^2/2}}{1+x}\right]^p w(x)\, dx < \infty$$

follows. Similar argument can be used to prove

$$\int_{-\infty}^0 \left[\frac{e^{x^2/2}}{1-x}\right]^p w(x)\, dx < \infty.$$

References

1. R. Askey, Mean convergence of orthogonal series and Lagrange interpolation, *Acta Math. Acad. Sci. Hungar.* **23** (1972), 71–85.
2. G. Freud, Lagrangesche Interpolation über die Nullstellen der Hermiteschen Orthogonalpolynome, *Studia Sci. Math. Hungar.* **4** (1968), 179–190.
3. G. Freud, On an inequality of Markov type, *Soviet Math Dokl.* **12** (1971), 570–573.
4. P. Nevai, Mean convergence of Lagrange interpolation, I, *J. Approx. Theory* **18** (1976), 363–377.
5. P. Nevai, Lagrange interpolation at zeros of orthogonal polynomials, *in* "Approximation Theory, II," pp. 163–201, Academic Press, New York, 1976.
6. P. Nevai, "Orthogonal Polynomials," Memoirs of the Amer. Math. Soc., Vol. 18, No. 213, 1979.
7. E. Stein, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, N. J., 1970.
8. G. Szegö, "Orthogonal Polynomials," Amer. Math. Soc., New York, 1967.